NASA TN D-1225

NASA TN D-1225

TECHNICAL NOTE

D-1225

VELOCITY POTENTIAL AND FORCES ON OSCILLATING SLENDER BODIES OF REVOLUTION IN SUPERSONIC FLOW EXPANDED TO THE FIFTH POWER OF THE FREQUENCY

By Donald L. Lansing

Langley Research Center
Langley Station, Hampton, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON April 1962

1F

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL NOTE D-1225

VELOCITY POTENTIAL AND FORCES ON OSCILLATING SLENDER BODIES OF REVOLUTION IN SUPERSONIC FLOW EXPANDED TO THE FIFTH POWER OF THE FREQUENCY

By Donald L. Lansing

L
1
1
9
5

SUMMARY

The linearized potential theory for supersonic flow about slender bodies of revolution undergoing harmonic oscillations is presented. The lift per unit length and the lift and moment coefficients on an arbitrary shape are expanded to the fifth power of the frequency. The theory is applied to a rigid cone undergoing harmonic translation and pitching motions. The results are compared with slender-body theory for the oscillating case and with Van Dyke's second-order theory for the steady case.

INTRODUCTION

The problem of determining the aerodynamic forces on bodies of revolution has been of considerable theoretical and experimental interest for some time. Until recently, the aerodynamics of lifting surfaces have been of more practical importance since the forces developed on an aircraft fuselage are usually small in comparison to those developed on wing surfaces. However, many of the vehicles designed over the past few years employ either very small-aspect-ratio lifting surfaces or none at all. The predominant air forces on such configurations are associated with the shape and flexibility of the body.

There are a number of theoretical methods for predicting the forces on bodies at supersonic speeds; the Mach number range and body shapes to which these methods are applicable vary considerably. For bodies fixed in the airstream, such approaches as linearized potential theory (refs. 1 and 2), second-order slender-body theory (refs. 3 and 4), shock-expansion theory (ref. 5), the tangent-cone approximation (ref. 6), Newtonian theory (refs. 6 and 7), methods based upon the use of complex variables (ref. 8), and the piston-theory approximation (ref. 9) are available. In unsteady flow the choice of methods is somewhat more limited. Among the techniques applicable to unsteady problems are the

Munk-Jones momentum theory (ref. 10), linearized potential theory (refs. 11 and 12), extended shock-expansion theory and a variational procedure (ref. 13), and various quasi-steady theories based upon steady-state results.

Some insight into the relative merits of the various methods applicable to unsteady flow has been provided by a recent experimental and analytical investigation of flutter of conical shells (ref. 14). In this study flutter calculations were made over a wide Mach number range and compared with experimental results. Potential theory, momentum theory, Newtonian theory, and several quasi-steady methods were used in making the calculations. The potential-theory aerodynamics employed in this investigation are derived herein.

The present paper is an extension of the previous work of reference 11 in which the velocity potential was expanded to the first power of the frequency. Another approach to calculating potential-theory aerodynamics on oscillating slender bodies was recently set forth in reference 12, which contains a description of a method for computing the pressures acting on the body by evaluating the potential and several of its derivatives by numerical processes. In the present work, the perturbation potential for an arbitrary slender body of revolution has been analytically expanded to the fifth power of the frequency. Expressions are given for the lift per unit length and the lift and moment coefficients. The coefficients of the powers of the frequency are obtained in the form of integrals involving the body shape and downwash. These general expressions are then evaluated for a rigid cone undergoing harmonic translation and pitching oscillations. Certain functions of Mach number and cone semiapex angle appearing in the expressions for the forces are tabulated. The results are compared with slender-body theory for the oscillating case and with Van Dyke's second-order theory for the steady case.

SYMBOLS

a_∞ free-stream speed of sound

$A_i^{(n)}$ functions of $\beta \tan \delta$ and M associated with lift per unit length on a cone (eqs. (A1))

$A^{(n)}(x)$ functions associated with lift per unit length on an arbitrary shape (eqs. (11a) to (11f))

b body length

C_L — total-lift coefficient (eqs. (13) and (14))

C_{M_α} — total-moment coefficient (eqs. (13) and (14))

$\bar{f}(x)$ — source or doublet strength

$f(x) = \frac{1}{2\pi} \bar{f}(x)$

L1195

h — amplitude of translation

k — reduced frequency, $\frac{\omega b}{2U_\infty}$

$l(x)$ — lift per unit length

L — total lift, (eq. (10a)) (positive down)

L_n — components of total-lift coefficient derived from potential theory (eq. (13))

$\bar{L}_n$ — components of total-lift coefficient derived from slender-body theory (eq. (14))

M — free-stream Mach number

M_{x_O} — total moment about $x = x_O$, (eq. (10b)) (positive, nose up)

M_n — components of total-moment coefficient derived from potential theory (eq. (13))

$\bar{M}_n$ — components of total-moment coefficient derived from slender-body theory (eq. (14))

p — perturbation pressure acting normal to body surface

P_n, Q_n — functions of $\beta \tan \delta$ associated with forces on a cone (eqs. (A2) and (A3))

q — free-stream dynamic pressure, $\frac{1}{2} \rho_\infty U_\infty^2$

$R(x)$ — body radius

$\bar{R}$ — polar radius of deforming body

$S(x)$ — body cross-sectional area, $\pi R^2(x)$

$S(b)$ — base area of cone, $\pi b^2 \tan^2 \delta$

t — time

U_∞ — free-stream velocity

x_0 — distance from nose to pitch axis

L1195

$\bar{x}_0 = \frac{x_0}{b}$

x,r,θ — a system of cylindrical coordinates with X-axis in direction of free stream and with origin located at mean position of body nose

$\bar{Z}(x,t)$ — downward displacement of body center line for arbitrary time-dependent motion

$Z(x)$ — amplitude of $\bar{Z}(x,t)$ for harmonic motion

$II_n(x,r)$ — integrals associated with the expanded velocity potential of an arbitrary body (eqs. (8a) to (8f))

$II_n(x) = \left[II_n(x,r)\right]_{r=R(x)}$

$II_n'(x) = \left[\frac{\partial}{\partial x} II_n(x,r)\right]_{r=R(x)}$

III_1, III_2, III_3 — integrals associated with velocity potential of an arbitrary body (eqs. (5a), (5b), and (5c))

α — angle of attack, or amplitude of pitch

$\beta = \sqrt{M^2 - 1}$

δ — cone semiapex angle

$\Delta(x)$ — function associated with perturbation pressure (eq. (9))

$\kappa = \frac{\omega}{\beta^2 a_\infty}$

ρ_∞ — free-stream density

$\bar{\varphi}(x,r,\theta,t)$ — perturbation velocity potential for an arbitrary time-dependent motion

$\varphi(x,r,\theta)$ — amplitude of perturbation potential for harmonic motion

$\varphi_1(x,r)$ — distribution of sources as defined in equation (4)

$\psi(x,r)$ — function associated with velocity potential (eq. (7))

ω — frequency of oscillation

L1195

ANALYSIS

The differential equation for the perturbation velocity potential is given and the solution appropriate to the lifting case is obtained. The solution involves a distribution function which is determined by applying the boundary condition of tangential velocity at the body surface. The potential, surface pressures, and lift per unit length are then expanded to the fifth power of the frequency. A more detailed derivation of the fundamental solution of the governing differential equation may be found in references 11 and 12.

The Governing Equation and the Fundamental Solution

The perturbation velocity potential $\bar{\varphi}(x,r,\theta,t)$ for the unsteady linearized supersonic flow about a body of revolution must satisfy the partial differential equation

$$\beta^2 \frac{\partial^2\bar{\varphi}}{\partial x^2} - \frac{\partial^2\bar{\varphi}}{\partial r^2} - \frac{1}{r}\frac{\partial\bar{\varphi}}{\partial r} - \frac{1}{r^2}\frac{\partial^2\bar{\varphi}}{\partial\theta^2} + \frac{1}{a_\infty^2}\frac{\partial^2\bar{\varphi}}{\partial t^2} + \frac{2M}{a_\infty}\frac{\partial^2\bar{\varphi}}{\partial x\,\partial t} = 0$$

where x,r,θ is a system of cylindrical coordinates moving with the flight velocity U_∞ in the negative x direction (see fig. 1(a)), a_∞ is the free-stream speed of sound, M is the free-stream Mach number, and $\beta^2 = M^2 - 1$. For harmonic motion, $\bar{\varphi}(x,r,\theta,t) = \varphi(x,r,\theta)e^{i\omega t}$ and the governing differential equation becomes

$$\beta^2 \frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial r^2} - \frac{1}{r}\frac{\partial\varphi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} - \frac{\omega^2}{a_\infty^2}\varphi + 2i\frac{M\omega}{a_\infty}\frac{\partial\varphi}{\partial x} = 0 \qquad (1)$$

In the linearized problem the perturbation potential is the sum of two types of potentials: a symmetric potential associated with body thickness which gives rise to no resultant lift or moment, and an antisymmetric potential associated with angle of attack, camber, and time-dependent motions which is used to determine total lift and moment. The remainder of this paper is concerned with the antisymmetric type of potential insofar as it applies to harmonic motions.

By direct substitution it may be verified that equation (1) is satisfied by the expression

$$\varphi(x,r,\theta) = \cos\theta \, \frac{\partial\varphi_1(x,r)}{\partial r} \tag{2}$$

L
1
1
9
5

where $\varphi_1(x,r)$ is a solution of the equation

$$\beta^2 \frac{\partial^2\varphi_1}{\partial x^2} - \frac{\partial^2\varphi_1}{\partial r^2} - \frac{1}{r}\frac{\partial\varphi_1}{\partial r} - \frac{\omega^2}{{a_\infty}^2}\varphi_1 + 2i\,\frac{M\omega}{a_\infty}\frac{\partial\varphi_1}{\partial x} = 0 \tag{3}$$

The form of the potential given in equation (2) is antisymmetric with respect to the plane $\theta = \pm\frac{\pi}{2}$, the x,y plane, and is therefore appropriate for determining lifting forces.

The solution of equation (3) corresponding to the potential φ_s of a moving point source of outgoing waves with frequency ω, is known to be

$$\varphi_s = -\frac{1}{2\pi}\frac{e^{-i\kappa Mx}\cos\left(\kappa\sqrt{x^2 - \beta^2 r^2}\right)}{\sqrt{x^2 - \beta^2 r^2}}$$

in which $\kappa = \frac{\omega}{\beta^2 a_\infty}$. The potential at a field point (x,r) resulting from a distribution of such sources along the X-axis of local strength $\bar{f}(x)dx$ is then

$$\varphi_1(x,r) = -\frac{1}{2\pi}\int_0^{x-\beta r}\bar{f}(\xi)\frac{e^{-iM\kappa(x-\xi)}\cos\left(\kappa\sqrt{(x-\xi)^2 - \beta^2 r^2}\right)}{\sqrt{(x-\xi)^2 - \beta^2 r^2}}\,d\xi \tag{4}$$

This potential is also a solution of equation (3). The upper limit of the integral indicates that only that portion of the line of sources within the upstream Mach cone with vertex at point x,r is taken into account.

Equations (2) and (4) will be used to determine the potential and forces on a lifting body of revolution. It is convenient to change from the variable ξ to the variable ζ by the substitution, $x - \xi = \beta r \cosh \zeta$, so that

$$\varphi_1(x,r) = -\int_0^{\cosh^{-1}\frac{x}{\beta r}} f(x - \beta r \cosh \zeta)\exp(-iM\kappa\beta r \cosh \zeta)\cos(\kappa\beta r \sinh \zeta)d\zeta$$

L
1
1
9
5

where $f(x) = \frac{1}{2\pi} \bar{f}(x)$. The differentiation indicated in equation (2), which converts the source distribution into a doublet distribution, may now be carried out yielding the desired form of the potential

$$\varphi(x,r,\theta) = \beta \cos \theta\left(iM\kappa III_1 + \kappa III_2 + III_3\right) \tag{5}$$

where

$$III_1 = \int_0^{\cosh^{-1}\frac{x}{\beta r}} f(x - \beta r \cosh \zeta)\exp(-iM\kappa\beta r \cosh \zeta)\cos(\kappa\beta r \sinh \zeta)\cosh \zeta \; d\zeta \tag{5a}$$

$$III_2 = \int_0^{\cosh^{-1}\frac{x}{\beta r}} f(x - \beta r \cosh \zeta)\exp(-iM\kappa\beta r \cosh \zeta)\sin(\kappa\beta r \sinh \zeta) \sinh \zeta \; d\zeta \tag{5b}$$

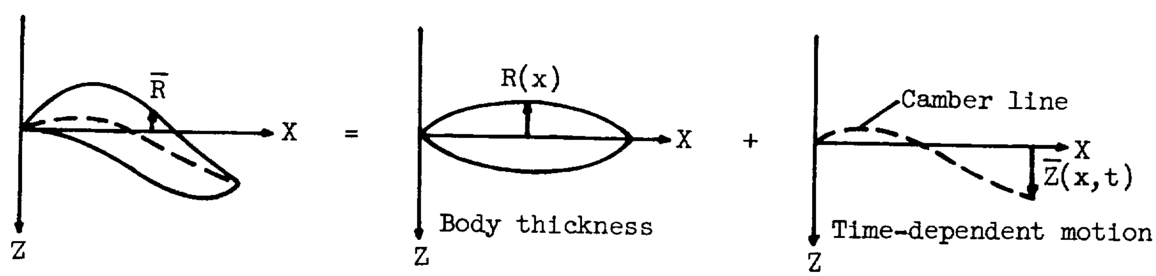
$$III_3 = \int_0^{\cosh^{-1}\frac{x}{\beta r}} f'(x - \beta r \cosh \zeta)\exp(-iM\kappa\beta r \cosh \zeta)\cos(\kappa\beta r \sinh \zeta) \cosh \zeta \; d\zeta \tag{5c}$$

and the prime denotes differentiation with respect to the argument. (In this and one further differentiation of $\varphi_1(x,r)$ to follow it is assumed that $f(0) = f'(0) = 0$. This assumption is consistent with the fact that, as will be shown subsequently, $f(x)$ is proportional to $R^2(x)$.)

The Determination of $f(x)$ From the Boundary Condition

The principal boundary condition to be satisfied is the requirement that the component of the fluid velocity normal to the body surface must vanish; that is, the flow must be tangent to the surface. Consider a slender body undergoing small time-dependent displacements about its equilibrium position. The shape of this body may be described as the superposition of a symmetric body of revolution, with local radius $R(x)$, about a mean camber line or center line which is undergoing small time-dependent displacements about its equilibrium position. See the following sketch:

L1195



Sketch 1

Thus, the deformation at any time may be considered as resulting from a vertical translation, with no rotation, of each vertical cross section of the body of revolution. Let $\bar{Z}(x,t)$ be the downward displacement of the center line of the body from the X-axis, and let $\bar{R}$ be the polar radius of the deforming body. See figure 1(b) and sketch 1. Applying the law of cosines to the triangle shown in figure 1(b) gives the following relation:

$$R^2(x) = \bar{Z}^2 + \bar{R}^2 + 2\bar{R}\bar{Z} \cos \theta$$

Solving for $\bar{R}$ and discarding the negative root results in

$$\bar{R} = \sqrt{R^2 - \bar{Z}^2 \sin^2\theta} - \bar{Z} \cos \theta$$

Approximating for small values of $\bar{Z}$ gives

$$\bar{R} = R(x) - \bar{Z}(x,t) \cos \theta$$

The linearized form of the boundary condition then requires the radial velocity on the mean body surface, $r = R(x)$, to be equal to the expression, $U_\infty \frac{\partial \bar{R}}{\partial x} + \frac{\partial \bar{R}}{\partial t}$, where U_∞ is the free-stream velocity. As in the case of the potential, the boundary condition may also be considered in two parts: one involving $R(x)$ associated with the thickness distribution, which is of no concern here, and the other involving $\bar{Z}(x,t)$ associated with the displacement. This latter part of the boundary condition takes the form

L
1
1
9
5

$$\lim_{r \to R(x)} \frac{\partial \bar{\varphi}}{\partial r} = -\cos \theta \left(U_\infty \frac{\partial \bar{Z}}{\partial x} + \frac{\partial \bar{Z}}{\partial t} \right)$$

For a body undergoing harmonic motions, let $\bar{Z}(x,t) = Z(x)e^{i\omega t}$. The boundary condition then becomes

$$\lim_{r \to R(x)} \frac{\partial \varphi}{\partial r} = -\cos \theta \left[U_\infty \frac{\partial Z(x)}{\partial x} + i\omega\, Z(x) \right]$$

The left-hand side of this expression can be evaluated by using equations (5). Carrying out the indicated differentiation, restoring the variable ξ by the substitution, $x - \xi = \beta r \cosh \zeta$, and approximating the integrals obtained for small values of r gives

$$\lim_{r \to R(x)} \frac{\partial \varphi}{\partial r} = -f(x)\frac{\cos \theta}{R^2(x)}$$

whence

$$f(x) = \frac{\bar{f}(x)}{2\pi} = R^2(x)\left[U_\infty \frac{\partial Z(x)}{\partial x} + i\omega\, Z(x) \right] \tag{6}$$

Expansion of the Potential and the Forces to the Fifth Power of the Frequency

Expanding the integrands of equations (5a), (5b), and (5c) in powers of κ and combining according to equation (5) results in the following expression to the fifth power in κ:

$$\varphi(x,r,\theta) = \beta \cos\theta \; \psi(x,r) \tag{7}$$

where

$$\begin{aligned}\psi(x,r) = {} & II_0(x,r) + iM\kappa \, II_1(x,r) + \frac{1}{2}\kappa^2\beta r \, II_2(x,r) \\ & - \frac{iM}{6}\kappa^3(\beta r)^2 \, II_3(x,r) - \frac{1}{24}\kappa^4(\beta r)^3 \, II_4(x,r) \\ & + \frac{iM}{120}\kappa^5(\beta r)^4 \, II_5(x,r)\end{aligned} \tag{8}$$

and

$$II_0(x,r) = \int_0^{\cosh^{-1}\frac{x}{\beta r}} f'(x - \beta r \cosh\zeta)\cosh\zeta \, d\zeta \tag{8a}$$

$$II_1(x,r) = -\beta r \int_0^{\cosh^{-1}\frac{x}{\beta r}} f'(x - \beta r \cosh\zeta) d\zeta \tag{8b}$$

$$\begin{aligned}II_2(x,r) = {} & -\beta r M^2 \int_0^{\cosh^{-1}\frac{x}{\beta r}} f'(x - \beta r \cosh\zeta)\cosh\zeta \, d\zeta \\ & + \left(M^2 - 1\right)\int_0^{\cosh^{-1}\frac{x}{\beta r}} f(x - \beta r \cosh\zeta) d\zeta\end{aligned} \tag{8c}$$

$$II_3(x,r) = -\beta r M^2 \int_0^{\cosh^{-1}\frac{x}{\beta r}} f'(x - \beta r \cosh \zeta)d\zeta$$

$$+ \left(M^2 - 3\right) \int_0^{\cosh^{-1}\frac{x}{\beta r}} f(x - \beta r \cosh \zeta)\cosh \zeta \, d\zeta \qquad (8d)$$

L
1
1
9
5

$$II_4(x,r) = -\beta r M^4 \int_0^{\cosh^{-1}\frac{x}{\beta r}} f'(x - \beta r \cosh \zeta)\cosh \zeta \, d\zeta$$

$$+ \left(M^4 + 3\right) \int_0^{\cosh^{-1}\frac{x}{\beta r}} f(x - \beta r \cosh \zeta)d\zeta$$

$$+ \left(M^4 - 6M^2 - 3\right) \int_0^{\cosh^{-1}\frac{x}{\beta r}} f(x - \beta r \cosh \zeta)\cosh^2\zeta \, d\zeta \qquad (8e)$$

$$II_5(x,r) = -\beta r M^4 \int_0^{\cosh^{-1}\frac{x}{\beta r}} f'(x - \beta r \cosh \zeta)d\zeta$$

$$+ \left(M^4 + 15\right) \int_0^{\cosh^{-1}\frac{x}{\beta r}} f(x - \beta r \cosh \zeta)\cosh \zeta \, d\zeta$$

$$+ \left(M^4 - 10M^2 - 15\right) \int_0^{\cosh^{-1}\frac{x}{\beta r}} f(x - \beta r \cosh \zeta)\cosh^3\zeta \, d\zeta \qquad (8f)$$

It may be noted that each of the equations (8a) to (8f), is a solution of equation (3).

The perturbation pressure acting normal to any point of the body surface is

$$p = -\rho_\infty \left(U_\infty \frac{\partial \varphi}{\partial x} + i\omega\varphi \right)_{r=R(x)}$$

Using equation (7) gives

$$p = -\rho_\infty U_\infty \beta \cos\theta \; \Delta(x) \tag{9}$$

L
1
1
9
5

where

$$\Delta(x) = \left[\frac{\partial \psi(x,r)}{\partial x} + i \frac{\omega}{U_\infty} \psi(x,r) \right]_{r=R(x)}$$

and the differentiation with respect to x is to be carried out before the substitution, $r = R(x)$, is made. The lifting component of this pressure is $p \cos\theta$; hence, the lift per unit length $l(x)$ is

$$\begin{aligned} l(x) &= R(x) \int_{-\pi}^{\pi} p \cos\theta \; d\theta \\ &= -\pi\rho_\infty U_\infty \beta \; R(x) \; \Delta(x) \end{aligned}$$

The total lift L and the total moment M_{x_0} about $x = x_0$ are then

$$L = \int_0^b l(x) \; dx = -\pi\rho_\infty U_\infty \beta \int_0^b R(x) \; \Delta(x) \; dx \tag{10a}$$

$$M_{x_0} = \int_0^b \left(x - x_0 \right) l(x) \; dx = -\pi\rho_\infty U_\infty \beta \int_0^b \left(x - x_0 \right) R(x) \; \Delta(x) \; dx \tag{10b}$$

With the aid of equation (8) in which κ is now replaced by $\frac{\omega}{U_\infty}\frac{M}{\beta^2}$ and equations (8a) to (8f), the expression for $\Delta(x)$ may be written as

$$\Delta(x) = A^{(0)}(x) + i\,\frac{\omega}{U_\infty}\,A^{(1)}(x) + \frac{1}{2}\left(\frac{\omega}{U_\infty}\right)^2 A^{(2)}(x) - \frac{i}{6}\left(\frac{\omega}{U_\infty}\right)^3 A^{(3)}(x)$$

L1195

$$- \frac{1}{24}\left(\frac{\omega}{U_\infty}\right)^4 A^{(4)}(x) + \frac{i}{120}\left(\frac{\omega}{U_\infty}\right)^5 A^{(5)}(x) \qquad (11)$$

where

$$A^{(0)}(x) = \Pi_0{}'(x) \qquad (11a)$$

$$A^{(1)}(x) = \frac{1}{\beta^2}\left[M^2\,\Pi_1{}'(x) + \beta^2\,\Pi_0(x)\right] \qquad (11b)$$

$$A^{(2)}(x) = \left(\frac{M}{\beta^2}\right)^2\left[\beta\,R(x)\,\Pi_2{}'(x) - 2\beta^2\,\Pi_1(x)\right] \qquad (11c)$$

$$A^{(3)}(x) = \frac{1}{\beta^2}\left(\frac{M}{\beta^2}\right)^2 \beta\,R(x)\left[M^2\beta\,R(x)\,\Pi_3{}'(x) - 3\beta^2\,\Pi_2(x)\right] \qquad (11d)$$

$$A^{(4)}(x) = \left(\frac{M}{\beta^2}\right)^4\left[\beta\,R(x)\right]^2\left[\beta\,R(x)\,\Pi_4{}'(x) - 4\beta^2\cdot\Pi_3(x)\right] \qquad (11e)$$

$$A^{(5)}(x) = \frac{1}{\beta^2}\left(\frac{M}{\beta^2}\right)^4\left[\beta\,R(x)\right]^3\left[M^2\beta\,R(x)\,\Pi_5{}'(x) - 5\beta^2\,\Pi_4(x)\right] \qquad (11f)$$

and

$$\Pi_n(x) = \left[\bar{\Pi}_n(x,r)\right]_{r=R(x)} \qquad \Pi_n{}'(x) = \left[\frac{\partial}{\partial x}\,\Pi_n(x,r)\right]_{r=R(x)}$$

The integrals, $II_n(x)$ and $II_n'(x)$, appearing in equations (11a) to (11f) must be evaluated for each particular body shape and deformation under consideration. If the displacement of the center line and the body radius can be represented by polynomials, the evaluation of these integrals in closed form is straightforward but tedious. The results obtained for $II_n(x)$ and $II_n'(x)$ involve expressions of the form $\sqrt{x^2 - \beta^2 R^2(x)}$ and $\cosh^{-1}\left[\frac{x}{\beta R(x)}\right]$. Thus, for simple body shapes and deformations, analytical procedures may be used to evaluate the lift and moment from equations (10).

L
1
1
9
5

Slender-Body Aerodynamics

A considerable simplification of the above results may be obtained by introducing the so-called "slender-body" approximation; that is, for smooth thin bodies the pressures acting on the body surface are essentially the same as those predicted on the X-axis. The expression for the lift per unit length on an arbitrary slender body will now be obtained as a limiting case of the equations developed previously. This result is subsequently used to evaluate the lift and moment coefficients on slender cones for comparison with the expansion procedure.

By making the substitution, $\beta r \cosh \zeta = x - \xi$, in equations (5a), (5b), and (5c) and approximating the three integrals for small values of r, the potential becomes

$$\varphi(x,r,\theta) = \frac{\cos\theta}{r} f(x)$$

which is now independent of the Mach number. As before,

$$f(x) = R^2(x)\left[U_\infty \frac{\partial Z(x)}{\partial x} + i\omega Z(x)\right]$$

The pertubation pressure acting normal to the body surface is

$$p = -\rho_\infty \frac{\cos\theta}{R(x)}\left[U_\infty \frac{\partial f(x)}{\partial x} + i\omega f(x)\right]$$

and the lift per unit length is

$$l(x) = R(x) \int_{-\pi}^{\pi} p \cos \theta \, d\theta = -2\pi q \left\{ \frac{\partial}{\partial x}\left[\frac{f(x)}{U_\infty}\right] + i\frac{\omega}{U_\infty}\left[\frac{f(x)}{U_\infty}\right] \right\}$$

For a body at a fixed angle of attack, with $\omega = 0$, and with $Z(x) = \alpha x$, the following well-known results of slender-body theory are obtained:

L1195

$$l(x) = -2\alpha q \frac{dS}{dx}$$

$$L = \int_0^b l(x)\, dx = -2\alpha q \, S(b) \tag{12a}$$

$$M_{x_0} = \int_0^b \left(x - x_0\right) l(x)\, dx = -2\alpha q b\left[\left(1 - \bar{x}_0\right) S(b) - \frac{V}{b}\right] \tag{12b}$$

where $S(x) = \pi \, R^2(x)$ is the cross-sectional area, $\bar{x}_0 = \frac{x_0}{b}$, and V is the volume of the body. A closed body, $S(b) = 0$, therefore develops no total lift, only a moment which tends to increase the angle of attack.

APPLICATION TO A CONE AND DISCUSSION

The expressions for the lift and moment developed previously for an arbitrary body of revolution have been evaluated to the fifth power of the frequency for a rigid cone undergoing harmonic pitch and translation. The final results are given in this section; further details in the derivation of the expressions presented here are given in the appendix. The corresponding expressions obtained from equations (12) have also been evaluated for the cone and are presented for comparison with the expansion procedure.

For a rigid body undergoing harmonic translation and pitch, the amplitude of the motion is $Z(x) = h + \alpha\left(x - x_0\right)$, see figure 2, where h and α are the amplitudes of translation and pitch, respectively, and x_0 is the distance of the pitch axis from the nose. In the following application, h is assumed positive down and α is positive when the nose is up. With these conventions, a positive lift acts downward and a positive moment acts nose upward.

Let the radius of the cone be $R(x) = x \tan \delta$, where δ is the semiapex angle, and let b be the length of the cone. Denote the base area of the cone, $\pi b^2 \tan^2 \delta$, by $S(b)$. The lift and moment coefficients are found to be

L 1 1 9 5

$$\left.\begin{aligned} C_L &= \frac{L}{q\ S(b)} = -4\beta \tan \delta \left[\left(L_1 + ikL_2\right)\frac{h}{b} + \left(L_3 + ikL_4\right)\alpha\right] \\ C_{M_\alpha} &= \frac{M_{x_0}}{qb\ S(b)} = -4\beta \tan \delta \left[\left(M_1 + ikM_2\right)\frac{h}{b} + \left(M_3 + ikM_4\right)\alpha\right] \end{aligned}\right\} \tag{13}$$

where q is the free-stream dynamic pressure, $\frac{1}{2} \rho_\infty U_\infty^2$, $k = \frac{b\omega}{2U_\infty}$, and

$$L_1 = -\frac{4}{3} A_0^{(1)} k^2 + \frac{8}{15} A_0^{(3)} k^4 \tag{13a}$$

$$L_2 = A_0^{(0)} + A_0^{(2)} k^2 - \frac{2}{9} A_0^{(4)} k^4 \tag{13b}$$

$$\begin{aligned} L_3 = \frac{1}{2} A_0^{(0)} &+ \left[\frac{1}{2}\left(A_0^{(2)} - A_1^{(1)}\right) + \frac{4}{3} \bar{x}_0 A_0^{(1)}\right] k^2 \\ &- \left[\frac{1}{9}\left(A_0^{(4)} - 2A_1^{(3)}\right) + \frac{8}{15} \bar{x}_0 A_0^{(3)}\right] k^4 \end{aligned} \tag{13c}$$

$$\begin{aligned} L_4 = &\left[\frac{1}{3}\left(2A_0^{(1)} + A_1^{(0)}\right) - \bar{x}_0 A_0^{(0)}\right] - \left[\frac{1}{15}\left(4A_0^{(3)} - 6A_1^{(2)}\right) + \bar{x}_0 A_0^{(2)}\right] k^2 \\ &+ \left[\frac{1}{105}\left(4A_0^{(5)} - 10A_1^{(4)}\right) + \frac{2}{9} \bar{x}_0 A_0^{(4)}\right] k^4 \end{aligned} \tag{13d}$$

$$M_1 = -2\left(\frac{1}{2} - \frac{2}{3}\bar{x}_0\right)A_0^{(1)}k^2 + 2\left(\frac{2}{9} - \frac{4}{15}\bar{x}_0\right)A_0^{(3)}k^4 \tag{13e}$$

$$M_2 = 2\left(\frac{1}{3} - \frac{\bar{x}_0}{2}\right)A_0^{(0)} + 2\left(\frac{2}{5} - \frac{\bar{x}_0}{2}\right)A_0^{(2)}k^2 - 2\left(\frac{2}{21} - \frac{\bar{x}_0}{9}\right)A_0^{(4)}k^4 \tag{13f}$$

L
1
1
9
5

$$M_3 = \left(\frac{1}{3} - \frac{\bar{x}_0}{2}\right)A_0^{(0)} + \left[\left(\frac{2}{5} - \frac{\bar{x}_0}{2}\right)\left(A_0^{(2)} - A_1^{(1)}\right) + 2\bar{x}_0\left(\frac{1}{2} - \frac{2\bar{x}_0}{3}\right)A_0^{(1)}\right]k^2$$
$$- \left[\left(\frac{2}{21} - \frac{\bar{x}_0}{9}\right)\left(A_0^{(4)} - 2A_1^{(3)}\right) + 2\bar{x}_0\left(\frac{2}{9} - \frac{4\bar{x}_0}{15}\right)A_0^{(3)}\right]k^4 \tag{13g}$$

$$M_4 = \left[\left(\frac{1}{4} - \frac{\bar{x}_0}{3}\right)\left(2A_0^{(1)} + A_1^{(0)}\right) - 2\bar{x}_0\left(\frac{1}{3} - \frac{\bar{x}_0}{2}\right)A_0^{(0)}\right]$$
$$- \left[\left(\frac{1}{9} - \frac{2\bar{x}_0}{15}\right)\left(2A_0^{(3)} - 3A_1^{(2)}\right) - 2\bar{x}_0\left(\frac{2}{5} - \frac{\bar{x}_0}{2}\right)A_0^{(2)}\right]k^2$$
$$+ \left[\left(\frac{1}{60} - \frac{2\bar{x}_0}{105}\right)\left(2A_0^{(5)} - 5A_1^{(4)}\right) + 2\bar{x}_0\left(\frac{2}{21} - \frac{\bar{x}_0}{9}\right)A_0^{(4)}\right]k^4 \tag{13h}$$

The quantities, $A_0^{(0)}, A_0^{(1)}, \ldots A_0^{(5)}$ and $A_1^{(0)}, A_1^{(1)}, \ldots A_1^{(5)}$, are functions of Mach number and the product, $\beta \tan \delta$. Tables I to V present values for various cone angles and Mach numbers. The calculations were carried out on an IBM 650 electronic data processing machine to eight significant figures. The values in the tables have been arbitrarily limited to six decimal places. The expressions from which $A_1^{(n)}$ may be evaluated are given in the appendix as equations (A1), (A2), and (A3).

For a cone undergoing rigid translation and pitch, slender-body theory (eqs. 12) gives

$$\left.\begin{aligned} C_L &= \frac{L}{q\ S(b)} = -4\left[\left(\bar{L}_1 + ik\bar{L}_2\right)\frac{h}{b} + \left(\bar{L}_3 + ik\bar{L}_4\right)\alpha\right] \\ C_{M_\alpha} &= \frac{M_{x_0}}{qb\ S(b)} = -4\left[\left(\bar{M}_1 + ik\bar{M}_2\right)\frac{h}{b} + \left(\bar{M}_3 + ik\bar{M}_4\right)\alpha\right] \end{aligned}\right\} \tag{14}$$

where L1195

$$\bar{L}_1 = -\frac{2}{3}k^2 \tag{14a}$$

$$\bar{L}_2 = 1 \tag{14b}$$

$$\bar{L}_3 = \frac{1}{2} - k^2\left(\frac{1}{2} - \frac{2}{3}\bar{x}_0\right) \tag{14c}$$

$$\bar{L}_4 = \frac{4}{3} - \bar{x}_0 \tag{14d}$$

$$\bar{M}_1 = -k^2\left(\frac{1}{2} - \frac{2}{3}\bar{x}_0\right) \tag{14e}$$

$$\bar{M}_2 = \frac{2}{3} - \bar{x}_0 \tag{14f}$$

$$\bar{M}_3 = \left(\frac{1}{3} - \frac{\bar{x}_0}{2}\right) - 2k^2\left(\frac{\bar{x}_0^{\,2}}{3} - \frac{\bar{x}_0}{2} + \frac{1}{5}\right) \tag{14g}$$

$$\bar{M}_4 = \left(1 - \bar{x}_0\right)^2 \tag{14h}$$

These results may also be obtained from equations (13) by using the following limiting forms for small δ:

$$A_0^{(0)} = \frac{1}{\beta \tan \delta} \qquad A_0^{(1)} = \frac{1}{2\beta \tan \delta} \qquad A_0^{(2)} = A_0^{(3)} = A_0^{(4)} = A_0^{(5)} = 0$$

$$A_1^{(0)} = \frac{3}{\beta \tan \delta} \qquad A_1^{(1)} = \frac{1}{\beta \tan \delta} \qquad A_1^{(2)} = A_1^{(3)} = A_1^{(4)} = A_1^{(5)} = 0$$

Figure 3 shows the ratios, $\beta \tan \delta \, L_n / \bar{L}_n$ and $\beta \tan \delta \, M_n / \bar{M}_n$, plotted against Mach number. These calculations are for $\delta = 7\frac{1}{2}^{\circ}$, $\bar{x}_0 = \frac{1}{2}$, and the two values of k, 0.05 and 0.10. It should be remembered that $\bar{L}_n$ and $\bar{M}_n$ are independent of Mach number and are therefore constant. The figures show that although L_2, L_3, M_2, and M_3 depend upon Mach number, the variation is small and the slender-body approximation for these quantities appears to be sufficiently accurate over a fairly wide Mach number range. On the other hand, L_1, L_4, M_1, and M_4 vary rapidly with Mach number and differ considerably from the corresponding expressions obtained from the slender-body approximation.

L1195

It should be pointed out that the eight ratios plotted in figure 3 tend to unity as $\delta \rightarrow 0^{\circ}$ but not as $M \rightarrow 1$. This behavior does not appear to be a limitation of the expansion procedure alone. The theory breaks down at very low supersonic Mach numbers since the potential (eq. (4)) becomes meaningless as $M \rightarrow 1$. For cones, it can be shown that the expansion of the integrands of the integrals in equations (5a), (5b), and (5c) to terms of the order k^5 is reliable for $\frac{k}{\beta^2} \leqq \frac{1}{2}$. Therefore, the lower limit of the Mach number range for which the theory is applicable is $M = \sqrt{1 + 2k}$.

The upper limit of the range of Mach numbers and cone angles to which the results obtained for the unsteady case are likely to be applicable can be determined by considering the results for the steady case. In particular, it may be pointed out that $A_1^{(n)} \rightarrow 0$ as $\beta \tan \delta \rightarrow 1$ so that the lift and the moment vanish. This is easily seen in the steady case, $k = 0$, for which the lift and moment coefficients given in equations (13) reduce to

$$\left.\begin{aligned} C_L &= -2\alpha\sqrt{1 - (\beta \tan \delta)^2} \\ C_{M_\alpha} &= -2\alpha\left(\frac{2}{3} - \bar{x}_0\right)\sqrt{1 - (\beta \tan \delta)^2} \end{aligned}\right\} \tag{15}$$

This behavior is readily explained on the basis of the mathematical solution of the problem by recalling that the only portion of the doublet distribution along the X-axis (eqs. (5)) which influences a point, $x, x \tan \delta$, on the cone surface lies within the upstream Mach cone with vertex at point $x, x \tan \delta$ and semiapex angle, $\tan^{-1} \frac{1}{\beta}$. When $\beta \tan \delta = 1$, the Mach cone and the conical body are tangent (along a ray passing through the vertex of each), and the effect of the doublet line has vanished. From a physical point of view, linear theory is restricted to Mach numbers for which the body lies well within the downstream Mach cone attached to the nose. This condition is obviously violated for $\beta \tan \delta \approx 1$. It should not be inferred, however, that the lift and moment for all body shapes decrease with increasing Mach number. For example, the present theory predicts an increase in lift with increasing Mach number on a semiparabolic body. (See refs. 11 and 12.)

L1195

Figure 4 also shows a comparison of the lift-curve slope of an inclined cone with $\delta = 7\frac{1}{2}^{\circ}$ as predicted by the present theory (eqs. (15)), slender-body theory (eq. (14)) with $k = 0$, and Van Dyke's second-order theory (ref. 3). Experimental measurements of forces on cones in steady flow (refs. 15 and 16) indicate that $C_L = -2\alpha$ is a useful approximation even for $\beta \tan \delta$ close to unity. Hence, on the basis of the behavior of equations (15) for the steady case, it may be expected that the results obtained in equations (13) for the unsteady case are limited to values of $\beta \tan \delta$ less than about one-half. In addition to Mach number and slenderness limitations, the frequency ω must be small so that the flow about the cone remains attached to the surface.

CONCLUDING REMARKS

The linearized potential theory for supersonic flow about slender bodies of revolution undergoing harmonic oscillations has been developed for the purpose of expanding the lift per unit length and the lift and moment coefficients on an arbitrary shape to the fifth power of the frequency. It is indicated that the expansion obtained is limited to a range of Mach numbers M and reduced frequencies k for which $\frac{k}{M^2 - 1} \leq \frac{1}{2}$. This parameter sets an upper limit to the frequency range and a lower limit to the Mach number range over which the results may be used.

The theory has been applied to a rigid cone undergoing translation and pitch, and the results are compared with slender-body theory for the oscillating case and with Van Dyke's second-order theory for the steady case. A comparison of the results obtained in the steady case with experimental measurements indicates that the theory is limited to Mach numbers M and cone semiapex angles δ for which $\sqrt{M^2 - 1}\tan\delta$ is less than about one-half. It has also been found that slender-body theory is not a consistently accurate approximation to potential theory.

L
1
1
9
5

Langley Research Center,
National Aeronautics and Space Administration,
Langley Air Force Base, Va., January 15, 1962.

APPENDIX

DERIVATION OF THE QUANTITIES $A_1^{(n)}$ ASSOCIATED WITH THE LIFT AND MOMENT ON A CONE UNDERGOING RIGID PITCH AND TRANSLATION

For the case of a cone undergoing rigid pitch and translation, the amplitude of the harmonic motion is $Z = h + \alpha\,(x - x_0)$ where h and α are the amplitudes of translation and pitch and x_0 is the pitch-axis location. The first step in obtaining $A_1^{(n)}$ is to evaluate the integrals $II_n(x)$ and $II_n'(x)$ with use of equations (8a) to (8f). To this end, it is convenient to write the expression, $U_\infty \frac{\partial Z}{\partial x} + i\omega Z$, appearing in equation (6) in the form

L
1
1
9
5

$$U_\infty \frac{\partial Z}{\partial x} + i\omega Z = U_\infty\left(W_0 + W_1 x\right)$$

where $W_0 = \alpha\left(1 - 2ik\bar{x}_0\right) + 2ik\,\frac{h}{b}$, $W_1 = \frac{2ik}{b}\,\alpha$, $k = \frac{\omega b}{2U_\infty}$, $\bar{x}_0 = \frac{x_0}{b}$, and b is the length of the cone. The distribution function $f(x)$ appearing in the integrals $II_n(x)$ and $II_n'(x)$ then becomes

$$f(x) = U_\infty R^2(x)\left(W_0 + W_1 x\right)$$

$$= U_\infty x^2 \tan^2\delta\left(W_0 + W_1 x\right)$$

It is now advantageous to split the distribution function $f(x)$ into two parts, $f(x) = f_0(x) + f_1(x)$ where $f_0(x) = U_\infty \tan^2\delta W_0 x^2$ and $f_1(x) = U_\infty \tan^2\delta W_1 x^3$, and to carry out the evaluation of the integrals and the functions $A^{(n)}(x)$ for each part separately. The following definitions are useful: $\eta = \beta \tan\delta$, $\overline{W}_0 = U_\infty W_0 \tan^2\delta$, $\overline{W}_1 = U_\infty W_1 \tan^2\delta$, $I_0 = \cosh^{-1}\frac{1}{\eta}$, and $I_1 = \frac{1}{\eta}\sqrt{1 - \eta^2}$.

Results for $f_0(x) = \overline{W}_0 x^2$

The integrals, $II_n(x)$ and $II_n'(x)$, are found to be:

$$II_0(x) = x\overline{W}_0\left(I_1 - \eta I_0\right)$$

$$II_0'(x) = 2\overline{W}_0 I_1$$

$$II_1(x) = -2\overline{W}_0 x^2 \eta\left(I_0 - \eta I_1\right)$$

L1195

$$II_1'(x) = -2\overline{W}_0 x \eta I_0$$

$$II_2(x) = x^2\overline{W}_0\left\{M^2\left[\left(1 + \frac{3}{2}\eta^2\right)I_0 - \frac{5}{2}\eta I_1\right] - \left(1 + \frac{1}{2}\eta^2\right)I_0 + \frac{3}{2}\eta I_1\right\}$$

$$II_2'(x) = 2x\overline{W}_0\left[M^2\left(I_0 - 2\eta I_1\right) - I_0 + \eta I_1\right]$$

$$II_3(x) = \overline{W}_0 x^2\left\{M^2\left[-3\eta I_0 + \left(\frac{1}{3} + \frac{8}{3}\eta^2\right)I_1\right] + 3\eta I_0 - \left(1 + 2\eta^2\right)I_1\right\}$$

$$II_3'(x) = x\overline{W}_0\left[M^2\left(-3\eta I_0 + I_1\right) + \left(3\eta I_0 - 3I_1\right)\right]$$

$$II_4(x) = \overline{W}_0 x^2\left\{M^4\left[\left(\frac{3}{2} + \frac{15}{8}\eta^2\right)I_0 + \left(\frac{1}{12\eta} - \frac{83}{24}\eta\right)I_1\right]\right.$$

$$+ M^2\left[-\left(3 + \frac{9}{4}\eta^2\right)I_0 + \left(-\frac{1}{2\eta} + \frac{23}{4}\eta\right)I_1\right]$$

$$\left. + \left(\frac{3}{2} + \frac{3}{8}\eta^2\right)I_0 + \left(-\frac{1}{4\eta} - \frac{13}{8}\eta\right)I_1\right\}$$

$$II_4'(x) = 2\overline{W}_0 x\left\{M^4\left[\frac{3}{2}I_0 + \left(\frac{1}{6\eta} - \frac{8}{3}\eta\right)I_1\right] + M^2\left[-3I_0 + \left(-\frac{1}{\eta} + 4\eta\right)I_1\right]\right.$$

$$\left. + \frac{3}{2}I_0 + \left(-\frac{1}{2\eta} - \eta\right)I_1\right\}$$

$$II_5'(x) = \bar{W}_0 x\left\{M^4\left[-\frac{15}{4}\eta I_0 + \left(\frac{1}{6\eta^2} + \frac{19}{12}\right)I_1\right] + M^2\left[\frac{15}{2}\eta I_0 - \left(\frac{5}{3\eta^2} + \frac{35}{6}\right)I_1\right]\right.$$

$$\left. - \frac{15}{4}\eta I_0 + \left(-\frac{5}{2\eta^2} + \frac{25}{4}\right)I_1\right\}$$

Substituting these values into equations (11a) to (11f) and rearranging gives

L1195

$$A^{(n)}(x) = 2\bar{W}_0 A_0^{(n)} x^n$$

where

$$A_0^{(0)} = P_0 \tag{A1a}$$

$$A_0^{(1)} = \frac{1}{\beta^2}\left(M^2 P_1 + P_2\right) \tag{A1b}$$

$$A_0^{(2)} = \left(\frac{M}{\beta^2}\right)^2\left(M^2 P_3 + P_4\right) \tag{A1c}$$

$$A_0^{(3)} = \frac{1}{\beta^2}\left(\frac{M}{\beta^2}\right)^2\left(M^4 P_5 + M^2 P_6 + P_7\right) \tag{A1d}$$

$$A_0^{(4)} = \left(\frac{M}{\beta^2}\right)^4\left(M^4 P_8 + M^4 P_9 + P_{10}\right) \tag{A1e}$$

$$A_0^{(5)} = \frac{1}{\beta^2}\left(\frac{M}{\beta^2}\right)^4\left(M^6 P_{11} + M^4 P_{12} + M^2 P_{13} + P_{14}\right) \tag{A1f}$$

and

$$P_0 = I_1 \tag{A2a}$$

$$P_1 = \frac{1}{2} I_1 - \frac{3}{2}\eta I_0 \tag{A2b}$$

$$P_2 = \frac{1}{2}\,\eta I_0 - \frac{1}{2}\,I_1 \tag{A2c}$$

$$P_3 = 3\eta I_0 - 4\eta^2 I_1 \tag{A2d}$$

$$P_4 = -3\eta I_0 + 3\eta^2 I_1 \tag{A2e}$$

L
1
1
9
5

$$P_5 = -\left(\frac{3}{2}\,\eta + \frac{15}{4}\,\eta^3\right)I_0 + \frac{17}{4}\,\eta^3 I_1 \tag{A2f}$$

$$P_6 = \left(3\eta + \frac{9}{2}\,\eta^3\right)I_0 - \frac{15}{2}\,\eta^2 I_1 \tag{A2g}$$

$$P_7 = -\left(\frac{3}{2}\,\eta + \frac{3}{4}\,\eta^3\right)I_0 + \frac{9}{4}\,\eta^2 I_1 \tag{A2h}$$

$$P_8 = \frac{15}{2}\,\eta^3 I_0 - \left(\frac{\eta^2}{2} + 8\eta^4\right)I_1 \tag{A2i}$$

$$P_9 = -15\eta^3 I_0 + \left(\frac{5}{3}\,\eta^2 + \frac{40}{3}\,\eta^4\right)I_1 \tag{A2j}$$

$$P_{10} = \frac{15}{2}\,\eta^3 I_0 - \left(\frac{5}{2}\,\eta^2 + 5\eta^4\right)I_1 \tag{A2k}$$

$$P_{11} = -\left(\frac{15}{4}\,\eta^3 + \frac{105}{16}\,\eta^5\right)I_0 + \left(-\frac{1}{8}\,\eta^2 + \frac{151}{16}\,\eta^4\right)I_1 \tag{A2l}$$

$$P_{12} = \left(\frac{45}{4}\,\eta^3 + \frac{225}{16}\,\eta^5\right)I_0 + \left(\frac{5}{8}\,\eta^2 - \frac{415}{16}\,\eta^4\right)I_1 \tag{A2m}$$

$$P_{13} = -\left(\frac{45}{4}\,\eta^3 + \frac{135}{16}\,\eta^5\right)I_0 + \left(-\frac{15}{8}\,\eta^2 + \frac{345}{16}\,\eta^4\right)I_1 \tag{A2n}$$

$$P_{14} = \left(\frac{15}{4}\,\eta^3 + \frac{15}{16}\,\eta^5\right)I_0 - \left(\frac{5}{8}\,\eta^2 + \frac{65}{16}\,\eta^4\right)I_1 \tag{A2o}$$

The lift and moment are then found from equations (10) and (11):

$$L = -4W_0 q\beta \tan\delta \; S(b)\left[\frac{A_0^{(0)}}{2} + ik\,\frac{2A_0^{(1)}}{3} + k^2\,\frac{A_0^{(2)}}{2} - ik^3\,\frac{4A_0^{(3)}}{15} - k^4\,\frac{A_0^{(4)}}{9} + ik^5\,\frac{4A_0^{(5)}}{105}\right]$$

$$M_{x_0} = -4W_0 qb\,\beta \tan\delta \; S(b)\left[\left(\frac{1}{3} - \frac{\bar{x}_0}{2}\right)A_0^{(0)} + ik\left(\frac{1}{2} - \frac{2}{3}\,\bar{x}_0\right)A_0^{(1)} + k^2\left(\frac{2}{5} - \frac{\bar{x}_0}{2}\right)A_0^{(2)} - ik^3\left(\frac{2}{9} - \frac{4\bar{x}_0}{15}\right)A_0^{(3)} - k^4\left(\frac{2}{21} - \frac{\bar{x}_0}{9}\right)A_0^{(4)} + ik^5\left(\frac{1}{30} - \frac{4\bar{x}_0}{105}\right)A_0^{(5)}\right]$$

Note that W_0 is of the form, $a + bk$. If the preceding two equations are multiplied out and rearranged in powers of k the resulting expression is correct only to k^5.

Results for $f_1(x) = \overline{W}_1 x^3$

The integrals, $II_n(x)$ and $II_n'(x)$, are found to be:

$$II_0 = 3\overline{W}_1 x^2\left[-\eta I_0 + \left(\frac{1}{3} + \frac{2}{3}\,\eta^2\right)I_1\right]$$

$$II_0' = 6\overline{W}_1 x\left(-\frac{\eta}{2}\,I_0 + \frac{1}{2}\,I_1\right)$$

$$II_1 = 3\overline{W}_1 x^3\left[-\left(\eta + \frac{1}{2}\,\eta^3\right)I_0 + \frac{3}{2}\,\eta^2 I_1\right]$$

$$II_1' = 6\bar{W}_1x^2\left(-\eta I_0 + \eta^2 I_1\right)$$

$$II_2 = \bar{W}_1x^3\left\{M^2\left[\left(1 + \frac{9}{2}\eta^2\right)I_0 - \left(\frac{17}{6}\eta + \frac{8}{3}\eta^3\right)I_1\right]\right.$$

$$\left. -\left(1 + \frac{3}{2}\eta^2\right)I_0 + \left(\frac{11}{6}\eta + \frac{2}{3}\eta^3\right)I_1\right\}$$

L1195

$$II_2' = 3\bar{W}_1x^2\left\{M^2\left[\left(1 + \frac{3}{2}\eta^2\right)I_0 - \frac{5}{2}\eta I_1\right] - \left(1 + \frac{\eta^2}{2}\right)I_0 + \frac{3}{2}\eta I_1\right\}$$

$$II_3 = \bar{W}_1x^3\left\{M^2\left[-\left(\frac{9}{2}\eta + \frac{15}{8}\eta^3\right)I_0 + \left(\frac{1}{4} + \frac{49}{8}\eta^2\right)I_1\right]\right.$$

$$\left. + \left(\frac{9}{2}\eta + \frac{9}{8}\eta^3\right)I_0 - \left(\frac{3}{4} + \frac{39}{8}\eta^2\right)I_1\right\}$$

$$II_3' = 3\bar{W}_1x^2\left\{M^2\left[-3\eta I_0 + \left(\frac{1}{3} + \frac{8}{3}\eta^2\right)I_1\right] + 3\eta I_0 - \left(1 + 2\eta^2\right)I_1\right\}$$

$$II_4 = \bar{W}_1x^3\left\{M^4\left[\left(\frac{3}{2} + \frac{45}{8}\eta^2\right)I_0 + \left(\frac{1}{20\eta} - \frac{477}{120}\eta - \frac{48}{15}\eta^3\right)I_1\right]\right.$$

$$+ M^2\left[-\left(3 + \frac{27}{4}\eta^2\right)I_0 + \left(-\frac{3}{10\eta} + \frac{137}{20}\eta + \frac{16}{5}\eta^3\right)I_1\right]$$

$$\left. + \left(\frac{3}{2} + \frac{9}{8}\eta^2\right)I_0 + \left(-\frac{3}{20\eta} - \frac{83}{40}\eta - \frac{2}{5}\eta^3\right)I_1\right\}$$

$$II_4' = 3\bar{W}_1x^2\left\{M^4\left[\left(\frac{3}{2} + \frac{15}{8}\eta^2\right)I_0 + \left(\frac{1}{12\eta} - \frac{83}{24}\eta\right)I_1\right]\right.$$

$$+ M^2\left[\left(-3 - \frac{9}{4}\eta^2\right)I_0 + \left(-\frac{1}{2\eta} + \frac{23}{4}\eta\right)I_1\right] + \left(\frac{3}{2} + \frac{3}{8}\eta^2\right)I_0$$

$$\left. + \left(-\frac{1}{4\eta} - \frac{13}{8}\eta\right)I_1\right\}$$

$$II_5' = 3\overline{W}_1 x^2 \left\{ M^4 \left[-\frac{15}{4}\eta I_0 + \left(\frac{1}{30\eta^2} + \frac{31}{60} + \frac{48}{15}\eta^2 \right) I_1 \right] + M^2 \left[\frac{15}{2}\eta I_0 - \left(\frac{1}{3\eta^2} + \frac{11}{6} + \frac{16}{3}\eta^2 \right) I_1 \right] - \frac{15}{4}\eta I_0 + \left(-\frac{1}{2\eta^2} + \frac{9}{4} + 2\eta^2 \right) I_1 \right\}$$

From equations (11a) to (11f)

L1195

$$A^{(n)}(x) = \overline{W}_1 A_1^{(n)} x^{n+1}$$

in which $A_1^{(n)}$ is defined as $A_0^{(n)}$ in equations (A1) with P_n replaced by Q_n, and

$$Q_0 = -3\eta I_0 + 3I_1 \tag{A3a}$$

$$Q_1 = -9\eta I_0 + \left(1 + 8\eta^2\right) I_1 \tag{A3b}$$

$$Q_2 = 3\eta I_0 - \left(1 + 2\eta^2\right) I_1 \tag{A3c}$$

$$Q_3 = \left(9\eta + \frac{15}{2}\eta^3\right) I_0 - \frac{33}{2}\eta^2 I_1 \tag{A3d}$$

$$Q_4 = \left(-9\eta - \frac{9}{2}\eta^3\right) I_0 + \frac{27}{2}\eta^2 I_1 \tag{A3e}$$

$$Q_5 = -\left(3\eta + \frac{45}{2}\eta^3\right) I_0 + \left(\frac{19}{2}\eta^2 + 16\eta^4\right) I_1 \tag{A3f}$$

$$Q_6 = \left(6\eta + 27\eta^3\right) I_0 - \left(17\eta^2 + 16\eta^4\right) I_1 \tag{A3g}$$

$$Q_7 = -\left(3\eta + \frac{9}{2}\eta^3\right) I_0 + \left(\frac{11}{3}\eta^2 + 2\eta^4\right) I_1 \tag{A3h}$$

$$Q_8 = \left(\frac{45}{2}\eta^3 + \frac{105}{8}\eta^5\right) I_0 - \left(\frac{3}{4}\eta^2 + \frac{279}{8}\eta^4\right) I_1 \tag{A3i}$$

$$Q_9 = -\left(45\eta^3 + \frac{75}{4}\eta^5\right)I_0 + \left(\frac{5}{2}\eta^2 + \frac{245}{4}\eta^4\right)I_1 \tag{A3j}$$

$$Q_{10} = \left(\frac{45}{2}\eta^3 + \frac{45}{8}\eta^5\right)I_0 - \left(\frac{15}{4}\eta^2 + \frac{195}{8}\eta^4\right)I_1 \tag{A3k}$$

$$Q_{11} = -\left(\frac{15}{2}\eta^3 + \frac{315}{8}\eta^5\right)I_0 + \left(-\frac{3}{20}\eta^2 + \frac{857}{40}\eta^4 + \frac{128}{5}\eta^6\right)I_1 \tag{A3l}$$

L
1
1
9
5

$$Q_{12} = \left(\frac{45}{2}\eta^3 + \frac{675}{8}\eta^5\right)I_0 + \left(\frac{3}{4}\eta^2 - \frac{477}{8}\eta^4 - 48\eta^6\right)I_1 \tag{A3m}$$

$$Q_{13} = -\left(\frac{45}{2}\eta^3 + \frac{405}{8}\eta^5\right)I_0 + \left(-\frac{9}{4}\eta^2 + \frac{411}{8}\eta^4 + 24\eta^6\right)I_1 \tag{A3n}$$

$$Q_{14} = \left(\frac{15}{2}\eta^3 + \frac{45}{8}\eta^5\right)I_0 - \left(\frac{3}{4}\eta^2 + \frac{83}{8}\eta^4 + 2\eta^6\right)I_1 \tag{A3o}$$

The lift and moment are

$$L = -2W_1 q\, \beta \tan\delta\, b\, S(b)\left[\frac{1}{3}A_1^{(0)} + ik\frac{A_1^{(1)}}{2} + k^2\frac{2A_1^{(2)}}{5} - ik^3\frac{2A_1^{(3)}}{9}\right.$$

$$\left. - k^4\frac{2A_1^{(4)}}{21} + ik^5\frac{A_1^{(5)}}{30}\right]$$

$$M_{x_0} = -2W_1 q\, \beta \tan\delta\, b^2\, S(b)\left[\left(\frac{1}{4} - \frac{\bar{x}_0}{3}\right)A_1^{(0)} + ik\left(\frac{2}{5} - \frac{\bar{x}_0}{2}\right)A_1^{(1)}\right.$$

$$+ k^2\left(\frac{1}{3} - \frac{2}{5}\bar{x}_0\right)A_1^{(2)} - ik^3\left(\frac{4}{21} - \frac{2}{9}\bar{x}_0\right)A_1^{(3)} - k^4\left(\frac{1}{12} - \frac{2}{21}\bar{x}_0\right)A_1^{(4)}$$

$$\left. + ik^5\left(\frac{4}{135} - \frac{\bar{x}_0}{30}\right)A_1^{(5)}\right]$$

REFERENCES

1. Liepmann, H. W., and Roshko, A.: Elements of Gasdynamics. John Wiley & Sons, Inc., c.1957, pp. 218-247.

2. Brown, Clinton E.: Aerodynamics of Bodies at High Speeds. Aerodynamic Components of Aircraft at High Speeds. Vol. VII of High Speed Aerodynamics and Jet Propulsion, sec. B, A. F. Donovan and H. R. Lawrence, eds., Princeton Univ. Press, 1957, pp. 244-280.

3. Van Dyke, Milton D.: A Study of Second-Order Supersonic Flow Theory. NACA Rep. 1081, 1952. (Supersedes NACA TN 2200.)

4. Van Dyke, Milton D.: Second-Order Slender-Body Theory - Axisymmetric Flow. NASA TR R-47, 1959. (Supersedes NACA TN 4281.)

5. Eggers, A. J., Jr., and Savin, Raymond C.: Approximate Methods for Calculating the Flow About Nonlifting Bodies of Revolution at High Supersonic Airspeeds. NACA TN 2579, 1951.

6. Lees, Lester: Hypersonic Flow. Fifth International Aeronautical Conference (Los Angeles, Calif., June 20-23, 1955), Inst. Aero. Sci., Inc., 1955, pp. 241-276.

7. Grimminger, G., Williams, E. P., Young, G. B. W.: Lift on Inclined Bodies of Revolution in Hypersonic Flow. Jour. Aero. Sci., vol. 17, no. 11, Nov. 1950, pp. 675-690.

8. Laporte, O., and Bartels, R. C. F.: An Investigation of the Exact Solutions of the Linearized Equations for the Flow Past Conical Bodies. Bumblebee Rep. No. 75 (Contract No. NOrd-7924), Eng. Res. Inst., Univ. of Michigan, Feb. 1948.

9. Miles, J. W., and Young, Dana: Generalized Missile Dynamics Analysis. III - Aerodynamics. GM-TR-0165-00360, Space Tech. Labs., The Ramo-Wooldridge Corp., Apr. 7, 1958.

10. Bisplinghoff, Raymond L., Ashley, Holt, and Halfman, Robert L.: Aeroelasticity. Addison-Wesley Pub. Co., Inc. (Cambridge, Mass.), c.1955, pp. 418-420.

11. Dorrance, William H.: Nonsteady Supersonic Flow About Pointed Bodies of Revolution. Jour. Aero. Sci., vol. 18, no. 8, Aug. 1951, pp. 505-511, 542.

12. Bond, Reuben, and Packard, Barbara B.: Unsteady Aerodynamic Forces on a Slender Body of Revolution in Supersonic Flow. NASA TN D-859, 1961.

L
1
1
9
5

13. Zartarian, Garabed, Hsu, Pao Tan, and Ashley, Holt: Dynamic Airloads and Aeroelastic Problems at Entry Mach Numbers. Jour. Aerospace Sci., vol. 28, no. 3, Mar. 1961, pp. 209-222.

14. Sewall, John L., Hess, Robert W., and Watkins, Charles E.: Analytical and Experimental Investigation of Flutter and Divergence of Spring-Mounted Cone Configurations at Supersonic Speeds. NASA TN D-1021, 1962.

15. Buford, William E.: The Effects of Afterbody Length and Mach Number on the Normal Force and Center of Pressure of Conical and Ogival Nose Bodies. Jour. Aero. Sci., vol. 25, no. 2, Feb. 1958, pp. 103-108.

16. Penland, Jim A.: Aerodynamic Force Characteristics of a Series of Lifting Cone and Cone-Cylinder Configurations at a Mach Number of 6.83 and Angles of Attack up to 130°. NASA TN D-840, 1961.

L
1
1
9
5

TABLE I.- VALUES OF THE QUANTITIES $A_i^{(n)}$

$[\delta = 5^\circ]$

$A_i^{(n)}$	Values of $A_i^{(n)}$ for Mach numbers of -								
	1.4	1.6	1.8	2.0	2.4	2.8	3.2	3.6	4.0
$A_0^{(0)}$	11.622803	9.096561	7.571268	6.522934	5.142629	4.254444	3.624800	3.150171	2.776641
$A_0^{(1)}$	5.125541	3.868857	3.091926	2.546291	1.807488	1.315630	.957243	.681532	.461800
$A_0^{(2)}$	.773693	.735021	.711635	.694958	.666933	.636708	.600670	.558177	.509594
$A_0^{(3)}$	-.120726	-.017855	.026165	.056237	.104228	.145373	.181138	.210965	.234053
$A_0^{(4)}$	-.411153	-.138398	-.073750	-.048102	-.021600	-.000237	.022292	.046474	.071524
$A_0^{(5)}$	-1.145841	-.267192	-.108560	-.061809	-.034631	-.024936	-.016565	-.006218	.006897
$A_1^{(0)}$	34.058872	26.337678	21.644602	18.398515	14.087906	11.284301	9.279181	7.757761	6.555734
$A_1^{(1)}$	8.724668	6.306740	4.795245	3.729733	2.294790	1.362217	.711253	.240584	-.104583
$A_1^{(2)}$	1.579430	1.461555	1.368088	1.287690	1.140615	.996105	.849625	.702145	.556245
$A_1^{(3)}$	-.133693	.059056	.135969	.184106	.250825	.296080	.323868	.335053	.330480
$A_1^{(4)}$	-.590626	-.178380	-.077704	-.035288	.010996	.046329	.078729	.107975	.132414
$A_1^{(5)}$	-1.367884	-.316266	-.124708	-.066800	-.029333	-.012121	.003407	.020250	.038242

L1195

L1195

TABLE II.- VALUES OF THE QUANTITIES $A_i^{(n)}$

$[\delta = 7.5^\circ]$

$A_i^{(n)}$	Values of $A_i^{(n)}$ for Mach numbers of -								
	1.4	1.6	1.8	2.0	2.4	2.8	3.2	3.6	4.0
$A_0^{(0)}$	7.687617	5.998690	4.975631	4.269874	3.334801	2.726722	2.290000	1.955514	1.687117
$A_0^{(1)}$	2.946478	2.122171	1.602780	1.232707	.724650	.383743	.136949	-.048651	-.190336
$A_0^{(2)}$	.845739	.781686	.730657	.685711	.599662	.509974	.414202	.313230	.209121
$A_0^{(3)}$	-.065101	.080837	.140086	.177589	.228455	.259153	.271314	.264534	.238765
$A_0^{(4)}$	-.576402	-.160352	-.055346	-.008602	.046752	.090020	.127290	.156196	.172922
$A_0^{(5)}$	-1.712327	-.391408	-.149539	-.073946	-.019119	.012644	.043267	.074427	.103209
$A_1^{(0)}$	22.003707	16.766977	13.562802	11.333247	8.350536	6.394929	4.989571	3.922113	3.081554
$A_1^{(1)}$	4.126418	2.657043	1.728265	1.076708	.226121	-.281324	-.586189	-.756619	-.831789
$A_1^{(2)}$	1.489451	1.318613	1.160310	1.015016	.748281	.504506	.284123	.091460	-.068858
$A_1^{(3)}$	-.047493	.210212	.295161	.334757	.359750	.341481	.291561	.218391	.130876
$A_1^{(4)}$	-.810475	-.187422	-.029635	.038986	.110629	.150762	.169130	.165706	.140968
$A_1^{(5)}$	-2.033182	-.451827	-.159066	-.064809	.007367	.046900	.076556	.096257	.102591

TABLE III.- VALUES OF THE QUANTITIES $A_i^{(n)}$

$[\delta = 10^{\circ}]$

$A_i^{(n)}$	Values of $A_i^{(n)}$ for Mach numbers of -								
	1.4	1.6	1.8	2.0	2.4	2.8	3.2	3.6	4.0
$A_0^{(0)}$	5.701190	4.429174	3.654952	3.117875	2.399380	1.924125	1.575084	1.299711	1.069686
$A_0^{(1)}$	1.778541	1.180128	.796321	.520255	.141047	-.107720	-.277621	-.391413	-.459946
$A_0^{(2)}$	.831885	.742519	.661713	.585949	.438294	.289941	.141871	-.001107	-.132067
$A_0^{(3)}$	-.006987	.171905	.234759	.266333	.288220	.271082	.219415	.137127	.030991
$A_0^{(4)}$	-.715276	-.154265	-.008761	.057523	.130815	.171649	.183520	.161078	.100691
$A_0^{(5)}$	-2.262971	-.498023	-.168417	-.059168	.031433	.085702	.124117	.139487	.119889
$A_1^{(0)}$	15.838195	11.838041	9.375552	7.653962	5.340962	3.822824	2.738287	1.926551	1.304143
$A_1^{(1)}$	1.693737	.767272	.186212	-.206065	-.664130	-.862094	-.902108	-.839052	-.708561
$A_1^{(2)}$	1.205351	1.003537	.800040	.611859	.282617	.016763	-.181027	-.306461	-.358545
$A_1^{(3)}$	-.012093	.286679	.354963	.361298	.300519	.190675	.061287	-.062941	-.159102
$A_1^{(4)}$	-.998980	-.172045	.031987	.112695	.169414	.162445	.112860	.035248	-.050449
$A_1^{(5)}$	-2.663537	-.560092	-.165346	-.034673	.063438	.099112	.098337	.064138	.004609

L1195

TABLE IV.- VALUES OF QUANTITIES $A_1^{(n)}$

$[\delta = 12.5^{\circ}]$

$A_1^{(n)}$	Values of $A_1^{(n)}$ for Mach numbers of -								
	1.4	1.6	1.8	2.0	2.4	2.8	3.2	3.6	4.0
$A_0^{(0)}$	4.493802	3.470248	2.843106	2.404613	1.809550	1.405216	1.096358	0.837383	0.597019
$A_0^{(1)}$	1.027886	.574625	.279966	.068053	-.216050	-.386460	-.479124	-.505992	-.463217
$A_0^{(2)}$	.755120	.642053	.531930	.426224	.222620	.029839	-.143084	-.279943	-.353361
$A_0^{(3)}$	.034565	.235016	.288208	.299058	.259770	.163636	.026482	-.129372	-.263677
$A_0^{(4)}$	-.833746	-.131128	.048835	.125170	.185051	.174915	.100267	-.032394	-.190963
$A_0^{(5)}$	-2.792810	-.582145	-.162832	-.019437	.097010	.140091	.121059	.026927	-.132503
$A_1^{(0)}$	12.042569	8.784655	6.769808	5.357788	3.462388	2.230952	1.372733	.760425	.332433
$A_1^{(1)}$	.201130	-.340840	-.663967	-.855409	-.995067	-.935609	-.761017	-.525962	-.276168
$A_1^{(2)}$	.808232	.603977	.384014	.185228	-.128839	-.323386	-.396720	-.357296	-.228711
$A_1^{(3)}$	-.056937	.265573	.301277	.262363	.111795	-.056236	-.186017	-.237136	-.188776
$A_1^{(4)}$	-1.177889	-.165041	.065558	.135980	.130796	.043182	-.069253	-.152568	-.155272
$A_1^{(5)}$	-3.259904	-.645226	-.156702	-.001302	.085190	.065558	-.008902	-.093971	-.127246

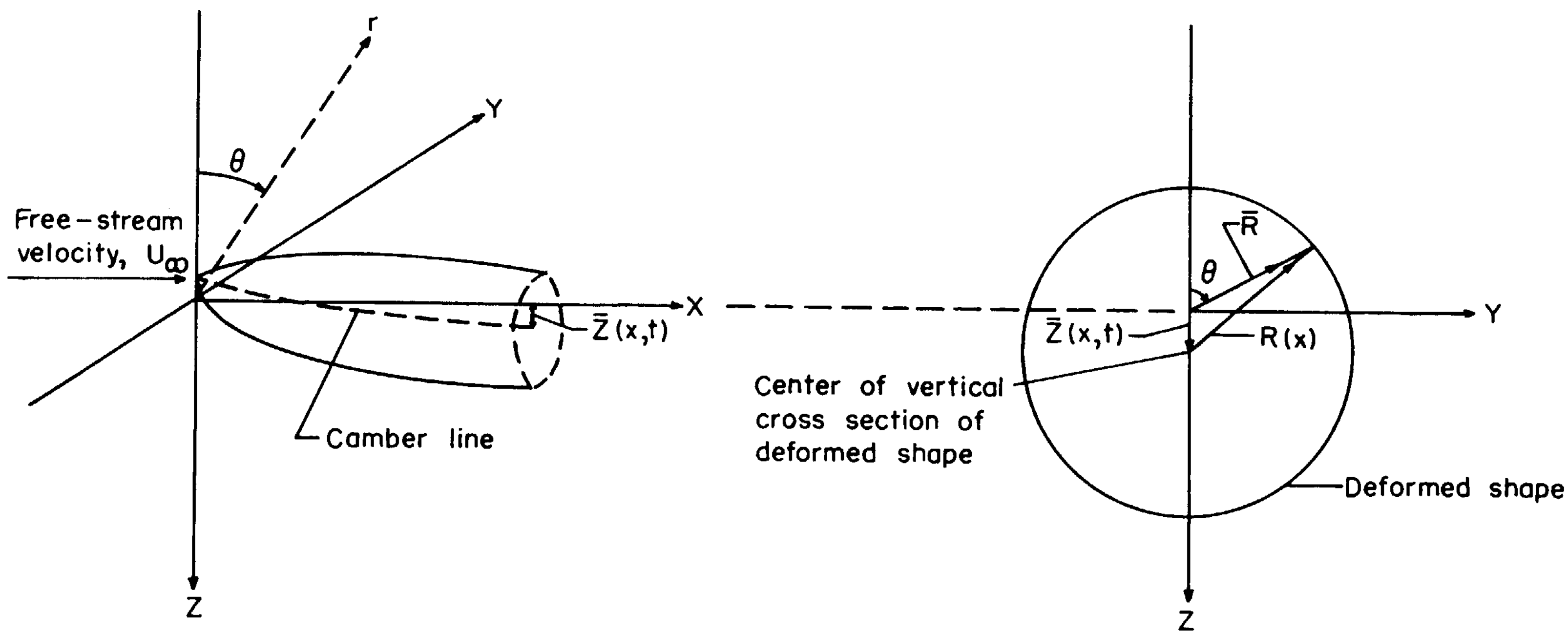
TABLE V.- VALUES OF QUANTITIES $A_i^{(n)}$

$[\delta = 15^{\circ}]$

$A_i^{(n)}$	Values of $A_i^{(n)}$ for Mach numbers[1] of -							
	1.4	1.6	1.8	2.0	2.4	2.8	3.2	3.6
$A_0^{(0)}$	3.675397	2.815730	2.284283	1.908595	1.387838	1.017982	0.712307	0.405667
$A_0^{(1)}$	.494587	.147836	-.078674	-.238594	-.436644	-.522382	-.511937	-.377242
$A_0^{(2)}$	.629089	.495441	.358920	.227468	-.016318	-.221465	-.356826	-.350652
$A_0^{(3)}$	.046103	.256282	.286300	.262394	.136900	-.047504	-.238110	-.325860
$A_0^{(4)}$	-.941818	-.106242	.095176	.163947	.162159	.044773	-.148630	-.302840
$A_0^{(5)}$	-3.297983	-.644860	-.140240	.028163	.129553	.085177	-.082642	-.281572
$A_1^{(0)}$	9.440880	6.681640	4.971224	3.773781	2.180167	1.174392	.517575	.118103
$A_1^{(1)}$	-.773372	-1.008521	-1.117629	-1.137823	-1.003309	-.732364	-.409023	-.113298
$A_1^{(2)}$	.349931	.176724	-.024748	-.195811	-.408837	-.443318	-.321870	-.108795
$A_1^{(3)}$	-.193665	.145078	.145632	.067371	-.127620	-.257926	-.252254	-.104588
$A_1^{(4)}$	-1.373221	-.197536	.037108	.076649	-.010757	-.142269	-.196999	-.100682
$A_1^{(5)}$	-3.826006	-.723741	-.160963	-.003366	.022209	-.073218	-.153501	-.097055

[1]When $M = 4.0$, $\beta \tan \delta > 1$.

L1195



(a) Cylindrical coordinates.

(b) Polar coordinates of deformed and undeformed body.

Figure 1.- Coordinate systems.

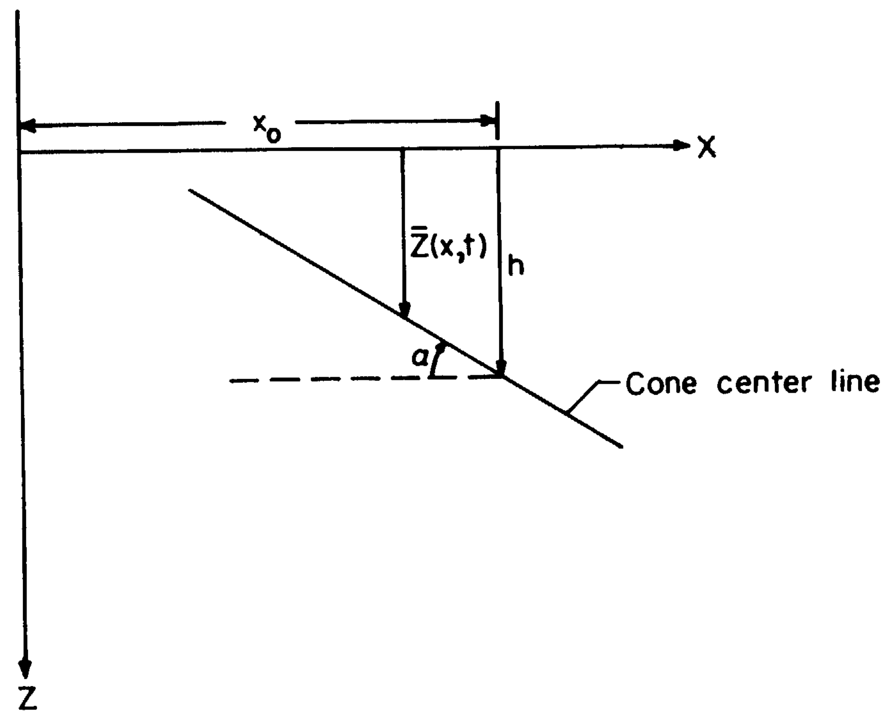
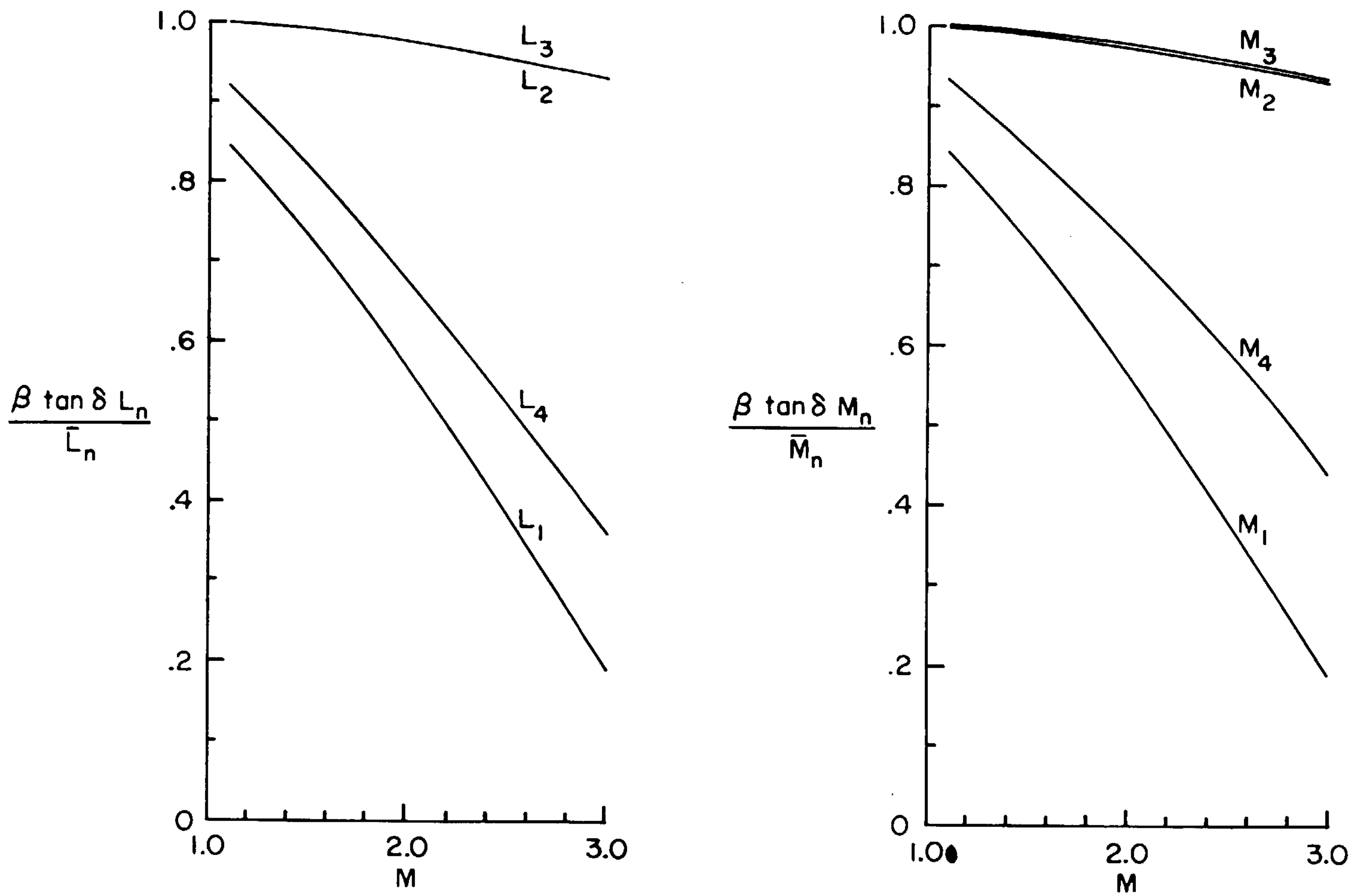


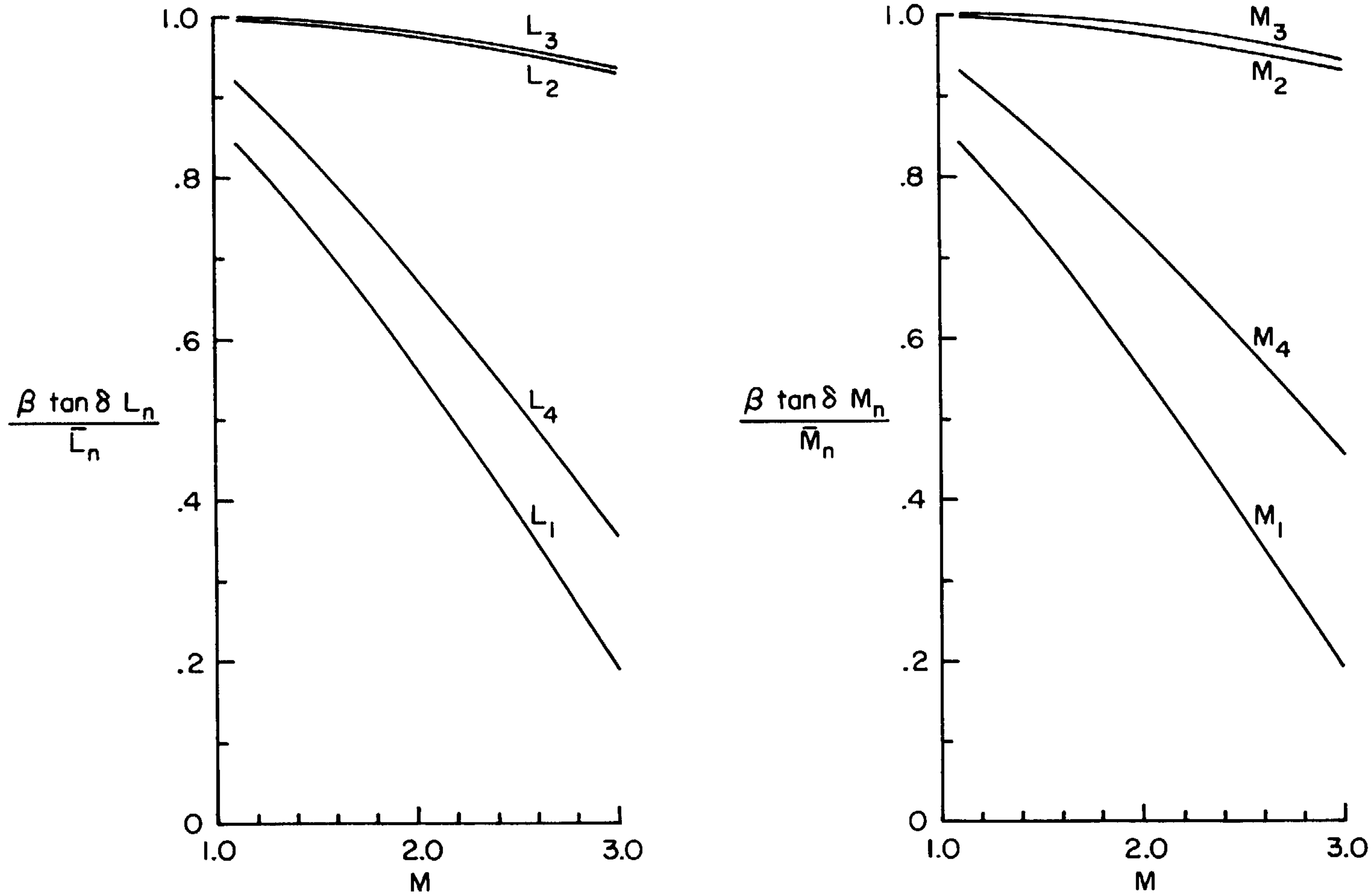
Figure 2.- Mode shape assumed for cone calculations.

$$\bar{Z}(x,t) = he^{i\omega t} + \alpha e^{i\omega t}\left(x - x_0\right).$$



(a) k = 0.05.

Figure 3.- Ratios of the lift and moment components of potential theory to the lift and moment components of slender-body theory for a cone with $\delta = 7\frac{1}{2}^{\circ}$ and $\bar{x}_0 = \frac{1}{2}$.



(b) k = 0.10.

Figure 3.- Concluded.

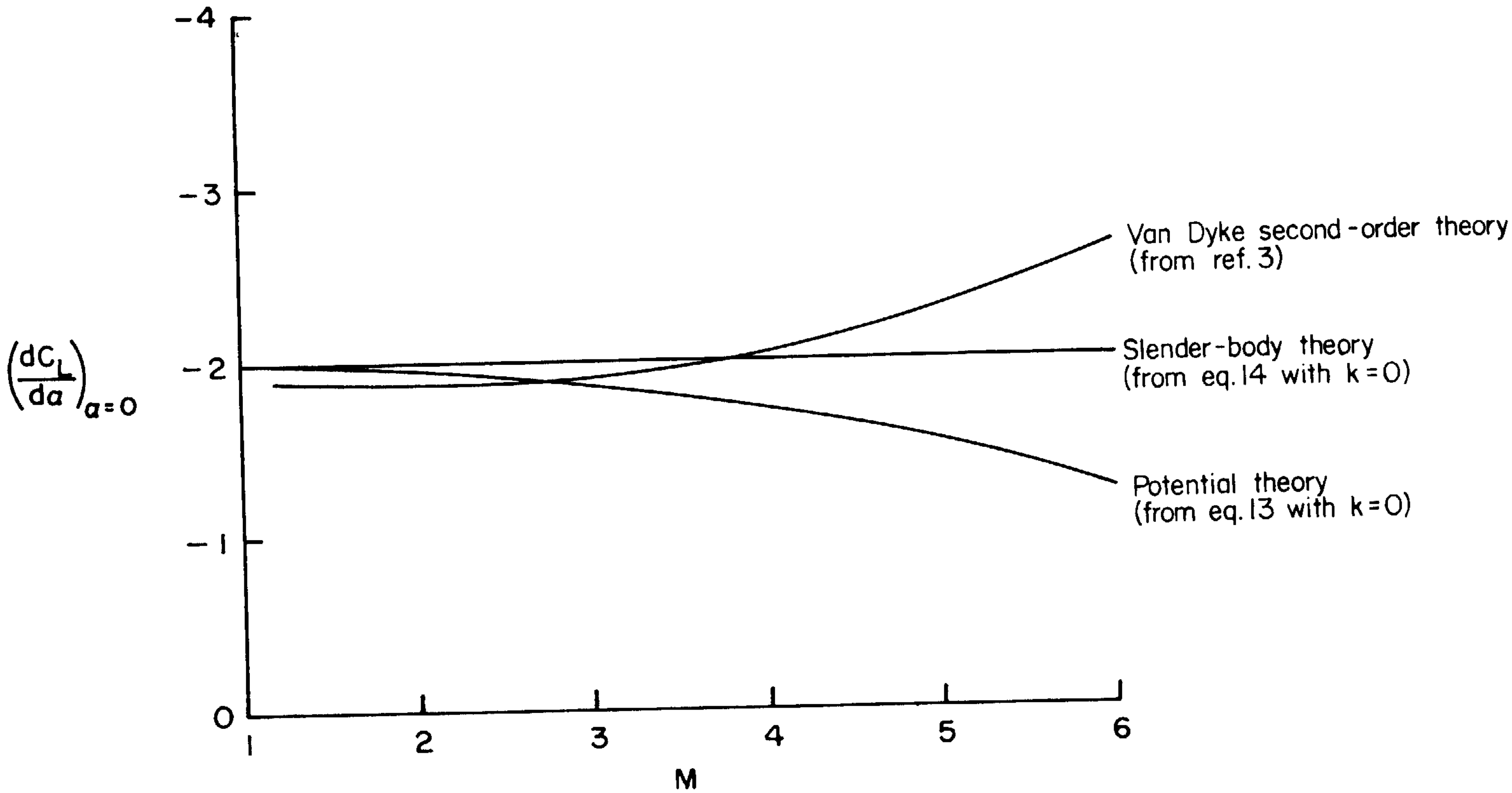


Figure 4.- Comparison of various solutions for the lift-curve slope of an inclined cone with a semiapex angle of $7\frac{1}{2}^{\circ}$.

NASA-Langley, 1962 L-1195